Problem with a solution proposed by Arkady Alt , **San Jose** , **California**, **USA.**

For any given positive integer $n \geq 3$ find smallest value of product $x_1x_2\ldots x_n$ where $x_1,x_2,\ldots,x_n > 0$ and $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\ldots+\frac{1}{1+x_n}=1.$

Solution.

Let $n=3$. We have $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}=1 \Leftrightarrow$

$3+2(x_1+x_2+x_3)+x_1x_2+x_2x_3+x_3x_1 = 1+x_1+x_2+x_3+x_1x_2+x_2x_3+x_3x_1+x_1x_2x_3 \Leftrightarrow 2+x_1+x_2+x_3 = x_1x_2x_3$. Since $x_1+x_2+x_3 \geq 3\sqrt[3]{x_1x_2x_3}$ then $x_1x_2x_3 \geq 2+3\sqrt[3]{x_1x_2x_3} \Leftrightarrow \left(\sqrt[3]{x_1x_2x_3}-2\right)\left(\sqrt[3]{x_1x_2x_3}+1\right)^2 \geq 0 \Leftrightarrow \sqrt[3]{x_1x_2x_3}-2 \geq 0 \Leftrightarrow x_1x_2x_3 \geq 2^3$.

Or another solution:

Since $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}=1 \Leftrightarrow \frac{1}{1+x_1}+\frac{1}{1+x_2}=\frac{x_3}{1+x_3} \Leftrightarrow$

$\frac{1+x_3}{1+x_1}+\frac{1+x_3}{1+x_2}=x_3 \Rightarrow x_3 \geq 2(1+x_3)\sqrt{\frac{1}{1+x_1}.\frac{1}{1+x_2}} = \frac{2(1+x_3)}{\sqrt{(1+x_1)(1+x_2)}}$.

Similarly we obtain $x_2 \geq \frac{2(1+x_2)}{\sqrt{(1+x_3)(1+x_1)}}, x_1 \geq \frac{2(1+x_1)}{\sqrt{(1+x_2)(1+x_3)}}$.

Hence, $x_1x_2x_3 \geq \frac{2^3(1+x_1)(1+x_2)(1+x_3)}{\sqrt{(1+x_2)(1+x_3)}\cdot\sqrt{(1+x_3)(1+x_1)}\cdot\sqrt{(1+x_1)(1+x_2)}} = 2^3$.

Using idea of this solution we can prove general case.

We have for any $k=1,2,\ldots,n$

$$\sum_{i=1}^{n}\frac{1}{1+x_i}=1 \Leftrightarrow \sum_{i=1,i\neq k}^{n}\frac{1}{1+x_i}=\frac{x_k}{1+x_k} \Leftrightarrow \sum_{i=1,i\neq k}^{n}\frac{1+x_k}{1+x_i}=x_k.$$

Then by AM-GM Inequality

$$x_k=\sum_{i=1,i\neq k}^{n}\frac{1+x_k}{1+x_i} \geq (n-1)\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}\frac{1+x_k}{1+x_i}} \Leftrightarrow x_k \geq \frac{(n-1)(1+x_k)}{\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}(1+x_i)}},\ k=1,2,\ldots,n.$$

Let $P := \prod_{k=1}^{n}(1+x_k)$. Since $\prod_{i=1,i\neq k}^{n}(1+x_i)=\frac{P}{1+x_k}$ then

$\prod_{k=1}^{n}\prod_{i=1,i\neq k}^{n}(1+x_i)=\frac{P^n}{\prod_{k=1}^{n}(1+x_k)}=P^{n-1}$ and, therefore,

$$\prod_{k=1}^{n}x_k \geq \prod_{k=1}^{n}\frac{(n-1)(1+x_k)}{\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}(1+x_i)}} = \frac{(n-1)^n\prod_{k=1}^{n}(1+x_k)}{\sqrt[n-1]{\prod_{k=1}^{n}\prod_{i=1,i\neq k}^{n}(1+x_i)}} = \frac{(n-1)^nP}{\sqrt[n-1]{P^{n-1}}} = (n-1)^n.$$